Introduction

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A logistic function or logistic curve is a common S-shaped curve (sigmoid curve) with the equation

$$f(x) = \frac{L}{1+e^{-k(x-x_0)}}$$

L → the supremum of the values of the function

x_0 → the x value of the function's midpoint

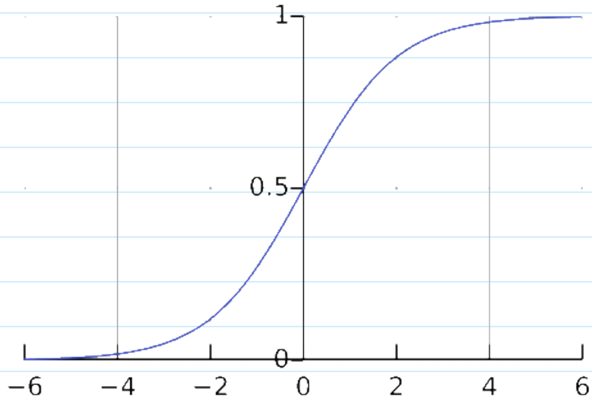
k → the steepness of the curve

$$f'(x) = \frac{-L \cdot e^{-k(x-x_0)} \cdot (-k)}{[1+e^{-k(x-x_0)}]^2} = \frac{Lk e^{-k(x-x_0)}}{[1+e^{-k(x-x_0)}]^2}$$

Since $e^x > 0$ for $x \in R$, the sign of $f'(x)$ is determined by the signs of L and k. Therefore $f(x)$ is either monotonically increasing or monotonically decreasing depending on the signs of L and k. $f(x)$ rearches its minimum 0 or maximum L as $x \to \pm\infty$.

The standard logistic function, where $L=1$, $k=1$, $x_0=0$ has the equation

$$f(x) = \frac{1}{1+e^{-x}}$$



Note: it is also sometimes called the sigmoid or expit, being the inverse function of the logit.

Mathematical properties

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$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1} = \frac{e^{x/2}}{e^{x/2}+e^{-x/2}}$$

Due to the nature of the exponential function e^{-x}, it can quickly converge to its saturation values 0 and 1.

1. Symmetry

$$f(x) + f(-x) = \frac{e^{x/2}}{e^{x/2}+e^{-x/2}} + \frac{e^{-x/2}}{e^{-x/2}+e^{x/2}} = 1$$

2. Inverse function

$$f(x) = \frac{1}{1+e^{-x}} \Rightarrow 1+e^{-x} = \frac{1}{f(x)} \Rightarrow e^{-x} = \frac{1-f(x)}{f(x)} \Rightarrow x = \ln\frac{f(x)}{1-f(x)} \text{ with } f(x) \in (0,1)$$

$$\xRightarrow{\text{let } p = f(x)} \operatorname{logit} p = \ln\frac{p}{1-p}, \quad p \in (0,1)$$

3. Derivative

$$f(x) = \frac{1}{1+e^{-x}} = \frac{e^x}{e^x+1} \Rightarrow f'(x) = \frac{e^x(e^x+1) - e^x \cdot e^x}{(1+e^x)^2} = \frac{e^x}{(1+e^x)^2} = \frac{e^x}{1+e^x} \cdot \frac{1}{1+e^x}$$

$$= \left(\frac{e^x}{1+e^x}\right)\left(1 - \frac{e^x}{1+e^x}\right) = f(x)[1-f(x)]$$

4. Integral

let $u = 1+e^x$, then we have

$$\left.\begin{aligned} f(x) &= \frac{e^x}{1+e^x} = \frac{u'}{u} \\ du &= e^x dx \end{aligned}\right\} \Rightarrow \int \frac{e^x}{1+e^x} dx = \int \frac{1}{u} du = \ln u = \ln(1+e^x)$$